

# A Double Cryptography Using The Keedwell Cross Inverse Quasigroup<sup>\*†</sup>

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## Abstract

The present study further strenghtens the use of the Keedwell CIPQ against attack on a system. This is done as follows. The holomorphic structure of AIPQs(AIPLs) and CIPQs(CIPLs) are investigated. Necessary and sufficient conditions for the holomorph of a quasigroup(loop) to be an AIPQ(AIPL) or CIPQ(CIPL) are established. It is shown that if the holomorph of a quasigroup(loop) is a AIPQ(AIPL) or CIPQ(CIPL), then the holomorph is isomorphic to the quasigroup(loop). Hence, the holomorph of a quasigroup(loop) is an AIPQ(AIPL) or CIPQ(CIPL) if and only if its automorphism group is trivial and the quasigroup(loop) is a AIPQ(AIPL) or CIPQ(CIPL). Furthermore, it is discovered that if the holomorph of a quasigroup(loop) is a CIPQ(CIPL), then the quasigroup(loop) is a flexible unipotent CIPQ(flexible CIPL of exponent 2). By constructing two isotopic quasigroups(loops)  $U$  and  $V$  such that their automorphism groups are not trivial, it is shown that  $U$  is a AIPQ or CIPQ(AIPL or CIPL) if and only if  $V$  is a AIPQ or CIPQ(AIPL or CIPL). Explanations and procedures are given on how these CIPQs can be used to double encrypt information.

## 1 Introduction

Let  $L$  be a non-empty set. Define a binary operation  $(\cdot)$  on  $L$  : If  $x \cdot y \in L$  for all  $x, y \in L$ ,  $(L, \cdot)$  is called a groupoid. If the system of equations ;

$$a \cdot x = b \quad \text{and} \quad y \cdot a = b$$

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have unique solutions for  $x$  and  $y$  respectively, then  $(L, \cdot)$  is called a quasigroup. For each  $x \in L$ , the elements  $x^\rho = xJ_\rho, x^\lambda = xJ_\lambda \in L$  such that  $xx^\rho = e^\rho$  and  $x^\lambda x = e^\lambda$  are called the right, left inverses of  $x$  respectively. Now, if there exists a unique element  $e \in L$  called the identity element such that for all  $x \in L$ ,  $x \cdot e = e \cdot x = x$ ,  $(L, \cdot)$  is called a loop. To every loop  $(L, \cdot)$  with automorphism group  $AUM(L, \cdot)$ , there corresponds another loop. Let the set  $H = (L, \cdot) \times AUM(L, \cdot)$ . If we define ' $\circ$ ' on  $H$  such that  $(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y)$  for all  $(\alpha, x), (\beta, y) \in H$ , then  $H(L, \cdot) = (H, \circ)$  is a loop as shown in Bruck [7] and is called the Holomorph of  $(L, \cdot)$ .

A loop(quasigroup) is a WIPL(WIPQ) if and only if it obeys the identity

$$x(yx)^\rho = y^\rho \quad \text{or} \quad (xy)^\lambda x = y^\lambda.$$

A loop(quasigroup) is a CIPL(CIPQ) if and only if it obeys the identity

$$xy \cdot x^\rho = y \quad \text{or} \quad x \cdot yx^\rho = y \quad \text{or} \quad x^\lambda \cdot (yx) = y \quad \text{or} \quad x^\lambda y \cdot x = y.$$

A loop(quasigroup) is an AIPL(AIPQ) if and only if it obeys the identity

$$(xy)^\rho = x^\rho y^\rho \text{ or } (xy)^\lambda = x^\lambda y^\lambda$$

Consider  $(G, \cdot)$  and  $(H, \circ)$  been two distinct groupoids(quasigroups, loops). Let  $A, B$  and  $C$  be three distinct non-equal bijective mappings, that maps  $G$  onto  $H$ . The triple  $\alpha = (A, B, C)$  is called an isotopism of  $(G, \cdot)$  onto  $(H, \circ)$  if and only if

$$xA \circ yB = (x \cdot y)C \quad \forall x, y \in G.$$

If  $(G, \cdot) = (H, \circ)$ , then the triple  $\alpha = (A, B, C)$  of bijections on  $(G, \cdot)$  is called an autotopism of the groupoid(quasigroup, loop)  $(G, \cdot)$ . Such triples form a group  $AUT(G, \cdot)$  called the autotopism group of  $(G, \cdot)$ . Furthermore, if  $A = B = C$ , then  $A$  is called an automorphism of the groupoid(quasigroup, loop)  $(G, \cdot)$ . Such bijections form a group  $AUM(G, \cdot)$  called the automorphism group of  $(G, \cdot)$ .

As observed by Osborn [15], a loop is a WIPL and an AIPL if and only if it is a CIPL. The past efforts of Artzy [2, 5, 4, 3], Belousov and Tzurkan [6] and present studies of Keedwell [11], Keedwell and Shcherbacov [12, 13, 14] are of great significance in the study of WIPLs, AIPLs, CIPQs and CIPLs, their generalizations(i.e m-inverse loops and quasigroups, (r,s,t)-inverse quasigroups) and applications to cryptography.

Interestingly, Adeniran [1] and Robinson [17], Oyebo and Adeniran [16], Chiboka and Solarin [9], Bruck [7], Bruck and Paige [8], Robinson [18], Huthnance [10] and Adeniran [1] have respectively studied the holomorphs of Bol loops, central loops, conjugacy closed loops, inverse property loops, A-loops, extra loops, weak inverse property loops, Osborn loops and Bruck loops. Huthnance [10] showed that if  $(L, \cdot)$  is a loop with holomorph  $(H, \circ)$ ,  $(L, \cdot)$  is a WIPL if and only if  $(H, \circ)$  is a WIPL. The holomorphs of an AIPL and a CIPL are yet to be studied.

In the quest for the application of CIPQs with long inverse cycles to cryptography, Keedwell [11] constructed the following CIPQ which we shall specifically call Keedwell CIPQ.

**Theorem 1.1** (*Keedwell CIPQ*) Let  $(G, \cdot)$  be an abelian group of order  $n$  such that  $n + 1$  is composite. Define a binary operation ' $\circ$ ' on the elements of  $G$  by the relation  $a \circ b = a^r b^s$ , where  $rs = n + 1$ . Then  $(G, \circ)$  is a CIPQ and the right crossed inverse of the element  $a$  is  $a^u$ , where  $u = (-r)^3$

The author also gave examples and detailed explanation and procedures of the use of this CIPQ for cryptography.

The aim of the present study is to further strengthen the use of the Keedwell CIPQ against attack on a system. This is done as follows.

1. The holomorphic structure of AIPQs(AIPLs) and CIPQs(CIPLs) are investigated. Necessary and sufficient conditions for the holomorph of a quasigroup(loop) to be an AIPQ(AIPL) or CIPQ(CIPL) are established. It is shown that if the holomorph of a quasigroup(loop) is a AIPQ(AIPL) or CIPQ(CIPL), then the holomorph is isomorphic to the quasigroup(loop). Hence, the holomorph of a quasigroup(loop) is an AIPQ(AIPL) or CIPQ(CIPL) if and only if its automorphism group is trivial and the quasigroup(loop) is a AIPQ(AIPL) or CIPQ(CIPL). Furthermore, it is discovered that if the holomorph of a quasigroup(loop) is a CIPQ(CIPL), then the quasigroup(loop) is a flexible unipotent CIPQ(flexible CIPL of exponent 2).
2. By constructing two isotopic quasigroups(loops)  $U$  and  $V$  such that their automorphism groups are not trivial, it is shown that  $U$  is a AIPQ or CIPQ(AIPL or CIPL) if and only if  $V$  is a AIPQ or CIPQ(AIPL or CIPL). Explanations and procedures are given on how these CIPQs can be used to double encrypt information.

## 2 Main Results

### 2.1 Holomorph Of AIPLs And CIPLs

**Theorem 2.1** Let  $(L, \cdot)$  be a quasigroup(loop) with holomorph  $H(L)$ .  $H(L)$  is an AIPQ(AIPL) if and only if

1.  $AUM(L)$  is an abelian group,
2.  $(\beta^{-1}, \alpha, I) \in AUT(L) \forall \alpha, \beta \in AUM(L)$  and
3.  $L$  is a AIPQ(AIPL).

#### Proof

A quasigroup(loop) is an automorphic inverse property loop(AIPL) if and only if it obeys the identity

$$(xy)^\rho = x^\rho y^\rho \text{ or } (xy)^\lambda = x^\lambda y^\lambda.$$

Using either of the definitions of an AIPQ(AIPL) above, it can be shown that  $H(L)$  is a AIPQ(AIPL) if and only if  $AUM(L)$  is an abelian group and  $(\beta^{-1}J_\rho, \alpha J_\rho, J_\rho) \in$

$AUT(L) \forall \alpha, \beta \in AUM(L)$ .  $L$  is isomorphic to a subquasigroup(subloop) of  $H(L)$ , so  $L$  is a AIPQ(AIPL) which implies  $(J_\rho, J_\rho, J_\rho) \in AUT(L)$ . So,  $(\beta^{-1}, \alpha, I) \in AUT(L) \forall \alpha, \beta \in AUM(L)$ .

**Corollary 2.1** *Let  $(L, \cdot)$  be a quasigroup(loop) with holomorph  $H(L)$ .  $H(L)$  is a CIPQ(CIPL) if and only if*

1.  $AUM(L)$  is an abelian group,
2.  $(\beta^{-1}, \alpha, I) \in AUT(L) \forall \alpha, \beta \in AUM(L)$  and
3.  $L$  is a CIPQ(CIPL).

**Proof**

A quasigroup(loop) is a CIPQ(CIPL) if and only if it is a WIPQ(WIPL) and an AIPQ(AIPL).  $L$  is a WIPQ(WIPL) if and only if  $H(L)$  is a WIPQ(WIPL).

If  $H(L)$  is a CIPQ(CIPL), then  $H(L)$  is both a WIPQ(WIPL) and a AIPQ(AIPL) which implies 1., 2., and 3. of Theorem 2.1. Hence,  $L$  is a CIPQ(CIPL). The converse follows by just doing the reverse.

**Corollary 2.2** *Let  $(L, \cdot)$  be a quasigroup(loop) with holomorph  $H(L)$ . If  $H(L)$  is an AIPQ(AIPL) or CIPQ(CIPL), then  $H(L) \cong L$ .*

**Proof**

By 2. of Theorem 2.1,  $(\beta^{-1}, \alpha, I) \in AUT(L) \forall \alpha, \beta \in AUM(L)$  implies  $x\beta^{-1} \cdot y\alpha = x \cdot y$  which means  $\alpha = \beta = I$  by substituting  $x = e$  and  $y = e$ . Thus,  $AUM(L) = \{I\}$  and so  $H(L) \cong L$ .

**Theorem 2.2** *The holomorph of a quasigroup(loop)  $L$  is a AIPQ(AIPL) or CIPQ(CIPL) if and only if  $AUM(L) = \{I\}$  and  $L$  is a AIPQ(AIPL) or CIPQ(CIPL).*

**Proof**

This is established using Theorem 2.1, Corollary 2.1 and Corollary 2.1.

**Theorem 2.3** *Let  $(L, \cdot)$  be a quasigroups(loop) with holomorph  $H(L)$ .  $H(L)$  is a CIPQ(CIPL) if and only if  $AUM(L)$  is an abelian group and any of the following is true for all  $x, y \in L$  and  $\alpha, \beta \in AUM(L)$ :*

1.  $(x\beta \cdot y)x^\rho = y\alpha$ .
2.  $x\beta \cdot yx^\rho = y\alpha$ .
3.  $(x^\lambda \alpha^{-1} \beta \alpha \cdot y\alpha) \cdot x = y$ .
4.  $x^\lambda \alpha^{-1} \beta \alpha \cdot (y\alpha \cdot x) = y$ .

**Proof**

This is achieved by simply using the four equivalent identities that define a CIPQ(CIPL):

$$xy \cdot x^\rho = y \quad \text{or} \quad x \cdot yx^\rho = y \quad \text{or} \quad x^\lambda \cdot (yx) = y \quad \text{or} \quad x^\lambda y \cdot x = y.$$

**Corollary 2.3** *Let  $(L, \cdot)$  be a quasigroup(loop) with holomorph  $H(L)$ . If  $H(L)$  is a CIPQ(CIPL) then the following are equivalent to each other*

1.  $(\beta^{-1}J_\rho, \alpha J_\rho, J_\rho) \in \text{AUT}(L) \forall \alpha, \beta \in \text{AUM}(L)$ .
2.  $(\beta^{-1}J_\lambda, \alpha J_\lambda, J_\lambda) \in \text{AUT}(L) \forall \alpha, \beta \in \text{AUM}(L)$ .
3.  $(x\beta \cdot y)x^\rho = y\alpha$ .
4.  $x\beta \cdot yx^\rho = y\alpha$ .
5.  $(x^\lambda \alpha^{-1} \beta \alpha \cdot y\alpha) \cdot x = y$ .
6.  $x^\lambda \alpha^{-1} \beta \alpha \cdot (y\alpha \cdot x) = y$ .

Hence,

$$(\beta, \alpha, I), (\alpha, \beta, I), (\beta, I, \alpha), (I, \alpha, \beta) \in \text{AUT}(L) \forall \alpha, \beta \in \text{AUM}(L).$$

**Proof**

The equivalence of the six conditions follows from Theorem 2.3 and the proof of Theorem 2.1. The last part is simply.

**Corollary 2.4** *Let  $(L, \cdot)$  be a quasigroup(loop) with holomorph  $H(L)$ . If  $H(L)$  is a CIPQ(CIPL) then,  $L$  is a flexible unipotent CIPQ(flexible CIPL of exponent 2).*

**Proof**

It is observed that  $J_\rho = J_\lambda = I$ . Hence, the conclusion follows.

**Example 2.1** *Let  $(L, \cdot)$  be an abelian group with  $\text{Inn}_\rho(L)$ -holomorph  $H(L)$ .  $H(L)$  is an abelian group.*

**Proof**

In an extra loop  $L$ ,  $\text{Inn}_\rho(L) = \text{Inn}_\lambda(L) \leq \text{AUM}(L)$  is a boolean group, hence it is abelian group. An abelian group is a commutative extra loop. A commutative extra loop is a CIPL. So by Corollary 2.1,  $H(L)$  is a CIPL.  $H(L)$  is a group since  $L$  is a group. A group is a CIPL if and only if it is abelian. Thus,  $H(L)$  is an abelian group.

**Remark 2.1** *The holomorphic structure of loops such as extra loop, Bol-loop, C-loop, CC-loop and A-loop have been found to be characterized by some special types of automorphisms such as*

1. Nuclear automorphism(in the case of Bol-,CC- and extra loops),
2. central automorphism(in the case of central and A-loops).

By Theorem 2.1 and Corollary 2.1, the holomorphic structure of AIPLs and CIPLs is characterized by commutative automorphisms. The abelian group in Example 2.1 is a boolean group.

## 2.2 A Pair Of AIPLs And CIPLs

**Theorem 2.4** *Let  $U = (L, \oplus)$  and  $V = (L, \otimes)$  be quasigroups such that  $AUM(U)$  and  $AUM(V)$  are conjugates in  $SYM(L)$  i.e there exists a  $\psi \in SYM(L)$  such that for any  $\gamma \in AUM(V)$ ,  $\gamma = \psi^{-1}\alpha\psi$  where  $\alpha \in AUM(U)$ . Then,  $H(U) \cong H(V)$  if and only if  $x\delta \otimes y\gamma = (x\beta \oplus y)\delta \forall x, y \in L, \beta \in AUM(U)$  and some  $\delta, \gamma \in AUM(V)$ . Hence:*

1.  $\gamma \in AUM(U)$  if and only if  $(I, \gamma, \delta) \in AUT(V)$ .
2. if  $U$  is a loop, then;
  - (a)  $\mathcal{L}_{e\delta} \in AUM(V)$ .
  - (b)  $\beta \in AUM(V)$  if and only if  $\mathcal{R}_{e\gamma} \in AUM(V)$ .

where  $e$  is the identity element in  $U$  and  $\mathcal{L}_x, \mathcal{R}_x$  are respectively the left and right translations mappings of  $x \in V$ .

3. if  $\delta = I$ , then  $|AUM(U)| = |AUM(V)| = 3$  and so  $AUM(U)$  and  $AUM(V)$  are boolean groups.
4. if  $\gamma = I$ , then  $|AUM(U)| = |AUM(V)| = 1$ .

### Proof

1. Let  $H(L, \oplus) = (H, \circ)$  and  $H(L, \otimes) = (H, \odot)$ .  $H(U) \cong H(V)$  if and only if there exists a bijection  $\phi : H(U) \rightarrow H(V)$  such that  $[(\alpha, x) \circ (\beta, y)]\phi = (\alpha, x)\phi \odot (\beta, y)\phi$ . Define  $(\alpha, x)\phi = (\psi^{-1}\alpha\psi, x\psi^{-1}\alpha\psi) \forall (\alpha, x) \in (H, \circ)$  where  $\psi \in SYM(L)$ .
2.  $H(U) \cong H(V) \Leftrightarrow (\alpha\beta, x\beta \oplus y)\phi = (\psi^{-1}\alpha\psi, x\psi^{-1}\alpha\psi) \odot (\psi^{-1}\beta\psi, y\psi^{-1}\beta\psi) \Leftrightarrow (\psi^{-1}\alpha\beta\psi, (x\beta \oplus y)\psi^{-1}\alpha\beta\psi) = (\psi^{-1}\alpha\beta\psi, x\psi^{-1}\alpha\beta\psi \otimes y\psi^{-1}\beta\psi) \Leftrightarrow (x\beta \oplus y)\psi^{-1}\alpha\beta\psi = x\psi^{-1}\alpha\beta\psi \otimes y\psi^{-1}\beta\psi \Leftrightarrow x\delta \otimes y\gamma = (x\beta \oplus y)\delta$  where  $\delta = \psi^{-1}\alpha\beta\psi, \gamma = \psi^{-1}\beta\psi$ .
3. Note that,  $\gamma\mathcal{L}_{x\delta} = L_{x\beta}\delta$  and  $\delta\mathcal{R}_{y\gamma} = \beta R_y\delta \forall x, y \in L$ . So, when  $U$  is a loop,  $\gamma\mathcal{L}_{e\delta} = \delta$  and  $\delta\mathcal{R}_{e\gamma} = \beta\delta$ . These can easily be used to prove the remaining part of the theorem.

**Theorem 2.5** *Let  $U = (L, \oplus)$  and  $V = (L, \otimes)$  be quasigroups(loops) that are isotopic under the triple of the form  $(\delta^{-1}\beta, \gamma^{-1}, \delta^{-1})$  for all  $\beta \in AUM(U)$  and some  $\delta, \gamma \in AUM(V)$  such that their automorphism groups are non-trivial and are conjugates in  $SYM(L)$  i.e there exists a  $\psi \in SYM(L)$  such that for any  $\gamma \in AUM(V)$ ,  $\gamma = \psi^{-1}\alpha\psi$  where  $\alpha \in AUM(U)$ . Then,  $U$  is a AIPQ or CIPQ(AIPL or CIPL) if and only if  $V$  is a AIPQ or CIPQ(AIPL or CIPL).*

### Proof

Let  $U$  be an AIPQ or CIPQ(AIPL or CIPL), then since  $H(U)$  has a subquasigroup(subloop) that is isomorphic to  $U$  and that subquasigroup(subloop) is isomorphic to a subquasigroup(subloop) of  $H(V)$  which is isomorphic to  $V$ ,  $V$  is a AIPQ or CIPQ(AIPL or CIPL). The proof for the converse is similar.

## 2.3 Application To Cryptography

Let the Keedwell CIPQ be the quasigroup  $U$  in Theorem 2.4. Definitely, its automorphism group is non-trivial because as shown in Theorem 2.1 of Keedwell [11], for any CIPQ, the mapping  $J_\rho : x \rightarrow x^\rho$  is an automorphism. This mapping will be trivial only if  $U$  is unipotent. For instance, in Example 2.1 of Keedwell [11], the CIPQ  $(G, \circ)$  obtained is unipotent because it was constructed using the cyclic group  $C_5 = \langle c : c^5 = e \rangle$  and defined as  $a \circ b = a^3 b^2$ . But in Example 2.2, the CIPQ gotten is not unipotent as a result of using the cyclic group  $C_{11} = \langle c : c^{11} = e \rangle$ . Thus the choice of a Keedwell CIPQ which suits our purpose in this work for a cyclic group of order  $n$  is one in which  $rs = n + 1$  and  $r + s \neq n$ . Now that we have seen a sample for the choice of  $U$ , the quasigroup  $V$  can then be obtained as shown in Theorem 2.4. By Theorem 2.5,  $V$  is a CIPQ.

In Keedwell [11], the author's method of application is as follows. It is assumed that the message to be transmitted can be represented as single element  $x$  of the quasigroup  $U$  and that this is enciphered by multiplying by another element  $y$  of  $U$  so that the encoded message is  $yx$ . At the receiving end, the message is deciphered by multiplying by the inverse of  $y$ . Now, according to Theorem 2.4, by the choice of the mappings  $\alpha, \beta \in AUM(U)$  and  $\psi \in Sym(L)$  to get the mappings  $\delta, \gamma$ , a CIPQ  $V$  can be produced following Theorem 2.4. So, the secret keys for the systems are  $\{\alpha, \beta, \psi\} \equiv \{\delta, \gamma\}$ . Thus whenever a set of information or messages is to be transmitted, the sender will encipher in the Keedwell CIPQ (as described earlier on) and then encipher again with  $\{\alpha, \beta, \psi\} \equiv \{\delta, \gamma\}$  to get a CIPQ  $V$  which is the set of encoded messages. At the receiving end, the message  $V$  is deciphered by using an inverse isotopism (i.e. inverse key  $\{\alpha, \beta, \psi\} \equiv \{\delta, \gamma\}$ ) to get  $U$  and then decipher again (as described earlier on) to get the messages. The secret key can be changed over time. The method described above is a double encryption and its a double protection. It protects each piece of information (element of the quasigroup) and protects the combined information (the quasigroup as a whole). Its like putting on a pair of socks and shoes or putting on under wears and clothes, the body gets better protection.

## References

- [1] J. O. Adeniran (2005), *On holomorphic theory of a class of left Bol loops*, Al.I.Cuza 51, 1, 23–28.
- [2] R. Artzy (1955), *On loops with special property*, Proc. Amer. Math. Soc. 6, 448–453.
- [3] R. Artzy (1959), *Crossed inverse and related loops*, Trans. Amer. Math. Soc. 91, 3, 480–492.
- [4] R. Artzy (1959), *On Automorphic-Inverse Properties in Loops*, Proc. Amer. Math. Soc. 10,4, 588–591.
- [5] R. Artzy (1978), *Inverse-Cycles in Weak-Inverse Loops*, Proc. Amer. Math. Soc. 68, 2, 132–134.

- [6] V. D. Belousov (1969), *Crossed inverse quasigroups(CI-quasigroups)*, Izv. Vyss. Ucebni; Zaved. Matematika 82, 21–27.
- [7] R. H. Bruck (1944), *Contributions to the theory of loops*, Trans. Amer. Math. Soc. 55, 245–354.
- [8] R. H. Bruck and L. J. Paige (1956), *Loops whose inner mappings are automorphisms*, The annuals of Mathematics, 63, 2, 308–323.
- [9] V. O. Chiboka and A. R. T. Solarin (1991), *Holomorphs of conjugacy closed loops*, Scientific Annals of Al.I.Cuza. Univ. 37, 3, 277–284.
- [10] E. D. Huthnance Jr.(1968), *A theory of generalised Moufang loops*, Ph.D. thesis, Georgia Institute of Technology.
- [11] A. D. Keedwell (1999), *Crossed-inverse quasigroups with long inverse cycles and applications to cryptography*, Australas. J. Combin. 20, 241-250.
- [12] A. D. Keedwell and V. A. Shcherbacov (2002), *On  $m$ -inverse loops and quasigroups with a long inverse cycle*, Australas. J. Combin. 26, 99-119.
- [13] A. D. Keedwell and V. A. Shcherbacov (2003), *Construction and properties of  $(r, s, t)$ -inverse quasigroups I*, Discrete Math. 266, 275-291.
- [14] A. D. Keedwell and V. A. Shcherbacov, *Construction and properties of  $(r, s, t)$ -inverse quasigroups II*, Discrete Math. 288 (2004), 61-71.
- [15] J. M. Osborn (1961), *Loops with the weak inverse property*, Pac. J. Math. 10, 295–304.
- [16] Y. T. Oyebo and O. J. Adeniran, *On the holomorph of central loops*, Pre-print.
- [17] D. A. Robinson (1964), *Bol loops*, Ph. D thesis, University of Wisconsin, Madison, Wisconsin.
- [18] D. A. Robinson (1971), *Holomorphic theory of extra loops*, Publ. Math. Debrecen 18, 59–64.